19/4/25

CHAPTER - 4
DETERMINANTS

★ DETERMINANT

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number called determinant of the square matrix A where $a_{ij} = (i, j)^{th}$ element of A.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(A)$

read as determinant of A

- **DETERMINANT OF A MATRIX OF ORDER ONE**

Let $A = [a]$ be the matrix of order 1,

then $|A| = a$.

- **DETERMINANT OF A MATRIX OF ORDER TWO**

$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Δ = Product of main diagonal elements − product of off diagonal elements

$= a_1 b_2 - a_2 b_1$

$\Delta = \begin{vmatrix} 3 & 4 \\ 9 & 7 \end{vmatrix}$

$= 3 \times 7 - 9 \times 4$

$= 21 - 36$

$= -15$

- **DETERMINANT OF A MATRIX OF ORDER THREE**

Determinant of a matrix of order three can be determined by expressing it in terms of second order determinants. This is known as expansion of a determinant along a row (or a column).

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

* <u>Expansion along first row (R_1)</u>

$$\underset{\substack{1+1=2 \\ \text{(even)} \\ +\text{ve sign}}}{a_{11}} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - \underset{\substack{1+2=3 \\ \text{(odd)} \\ -\text{ve sign}}}{a_{12}} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + \underset{\substack{1+3=4 \\ \text{(even)} \\ +\text{ve sign}}}{a_{13}} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

* <u>Expansion along second column (C_2)</u>

$$-\underset{\substack{1+2=3 \text{ (odd)} \\ -\text{ve sign}}}{a_{12}} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + \underset{\substack{2+2=4 \\ \text{(even)} \\ +\text{ve sign}}}{a_{22}} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - \underset{\substack{3+2=5 \\ \text{(odd)} \\ -\text{ve sign}}}{a_{32}} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

- SINGULAR AND NON-SINGULAR MATRIX

Let A be a square matrix such that $|A| = 0$, then it is called singular matrix.

If $|A| \neq 0$, then it is called non-singular matrix.

- If A is a square matrix of order n, then

$$|kA| = k^n |A|$$

Let A is a square matrix of order 3

$$|A| = 10$$

$$|3A| = 3^3 |A|$$
$$= 27 \times 10$$
$$= 270$$

★ MINORS AND COFACTORS

- Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its row and jth column in which element a_{ij} lies. Minor of an element is denoted by M_{ij}.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

- Cofactor of an element a_{ij}, denoted by A_{ij} or C_{ij} is defined by
  $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is minor of a_{ij}

$$A_{11} = (-1)^{1+1} M_{11} = M_{11}$$
$$A_{12} = (-1)^{1+2} M_{12} = -M_{12}$$

* $\Delta = M_{11}C_{11} + M_{12}C_{12} + M_{13}C_{13}$

* $\Delta = 0$

when $\Delta = M_{11}C_{21} + M_{12}C_{22} + M_{13}C_{23}$

★ AREA OF TRIANGLE

(x_1, y_1), (x_2, y_2), (x_3, y_3)

* Area of $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

~~Area of Δ =~~

* Equation of line passing through (x_1, y_1) and (x_2, y_2)

$$= \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$$

* If points are collinear,

$$\Delta = 0$$

ADJOINT OF A MATRIX

The adjoint of a square matrix $A = [a_{ij}]_{n\times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n\times n}$, where A_{ij} is the cofactor of the element a_{ij}.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$$

$$A_{11} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 \qquad A_{12} = -\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -12 \qquad A_{13} = \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 6$$

$$A_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \qquad A_{22} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 5 \qquad A_{23} = -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = 2$$

$$A_{31} = \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -11 \qquad A_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -1 \qquad A_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5$$

$$\text{adj } A = \begin{bmatrix} 3 & -12 & 6 \\ 1 & 5 & 2 \\ -11 & -1 & 5 \end{bmatrix}^t = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$A_{11} = \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 7 \qquad A_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1 \qquad A_{13} = \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = -1$$

$A_{21} = -\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = -3 \qquad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1 \qquad A_{23} = -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$

$A_{31} = \begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = -3 \qquad A_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0 \qquad A_{33} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$

$$\text{adj } A = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^t = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

21/4/23

★ EXERCISE - 4.1

- Evaluate the determinants in Exercises 1 and 2.

Q1.) $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

Sol. $\Delta \Rightarrow -2 - (-5 \times 4) \Rightarrow -2 - (-20) \rightarrow 18$ Ans

---

Q2.(i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

(i) $\Delta \Rightarrow \cos^2\theta - (-\sin^2\theta) \Rightarrow \cos^2\theta + \sin^2\theta = 1$ Ans

(ii) $\begin{vmatrix} x^2 - x + 1 & x-1 \\ x+1 & x+1 \end{vmatrix}$

(ii) $[(x^2 - x + 1)(x+1)] - [(x+1)(x-1)]$

$\Rightarrow (x^3 + x^2 - x^2 + x + 1) - (x^2 - 1)$

$\Rightarrow x^3 + 1 - x^2 + 1$

$\Rightarrow x^3 - x^2 + 2$ Ans

Q3.) If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$

Sol. $\left| 2\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} \right| = 4 \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$

$\Rightarrow \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix} = 4(2-8)$

$\Rightarrow 8 - 32 = \cancel{4(2-8)}\ 4(-6)$

$\Rightarrow -24 = -24$

L.H.S. = R.H.S

Hence, proved.

Q4.) If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then show that $|3A| = 27|A|$

Sol. L.H.S

$$|3A| = \begin{vmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{vmatrix}$$

Expanding along C_1;

$$= 3\begin{vmatrix} 3 & 6 \\ 0 & 12 \end{vmatrix} - 0\begin{vmatrix} 0 & 3 \\ 0 & 12 \end{vmatrix} + 0\begin{vmatrix} 0 & 3 \\ 3 & 6 \end{vmatrix}$$

$$= 3(36-0) - 0 + 0$$

$$= 108$$

R.H.S.

$$27|A| = 27\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix}$$

Expanding along C_1,

$$= 27\left[1\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 0 & 4 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}\right]$$

$$= 27[(4-0) - 0 + 0]$$

$$= 27 \times 4$$

$$= 108$$

L.H.S. = R.H.S.

Hence, proved

Q5.) Evaluate the determinants.

(i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

(i) Expanding along R_2,

$-0\begin{vmatrix}-1 & -2\\ -5 & 0\end{vmatrix} + 0\begin{vmatrix}3 & -2\\ 3 & 0\end{vmatrix} - (-1)\begin{vmatrix}3 & -1\\ 3 & -5\end{vmatrix}$

$\Rightarrow -0+0+1[-15-(-3)]$

$\Rightarrow -15+3$

$\Rightarrow -12$ Ans

ii) $\begin{vmatrix}3 & -4 & 5\\ 1 & 1 & -2\\ 2 & 3 & 1\end{vmatrix}$

ii) Exanding along R_1,

$3\begin{vmatrix}1 & -2\\ 3 & 1\end{vmatrix} - (-4)\begin{vmatrix}1 & -2\\ 2 & 1\end{vmatrix} + 5\begin{vmatrix}1 & 1\\ 2 & 3\end{vmatrix}$

$\Rightarrow 3(1+6)+4(1+4)+5(3-2)$

$\Rightarrow 21+20+5$

$\Rightarrow 46$ Ans

iii) $\begin{vmatrix}0 & 1 & 2\\ -1 & 0 & -3\\ -2 & 3 & 0\end{vmatrix}$

iii) Expanding along R_1,

$0\begin{vmatrix}0 & -3\\ 3 & 0\end{vmatrix} - 1\begin{vmatrix}-1 & -3\\ -2 & 0\end{vmatrix} + 2\begin{vmatrix}-1 & 0\\ -2 & 3\end{vmatrix}$

$\Rightarrow 0-1(-0-6)+2(-3-0)$

$\Rightarrow 6-6$

$\Rightarrow 0$ Ans

iv) $\begin{vmatrix}2 & -1 & -2\\ 0 & 2 & -1\\ 3 & -5 & 0\end{vmatrix}$

iv) Expanding along C_1,

$= 2\begin{vmatrix}2 & -1\\ -5 & 0\end{vmatrix} - 0\begin{vmatrix}-1 & -2\\ -5 & 0\end{vmatrix} + 3\begin{vmatrix}-1 & -2\\ 2 & -1\end{vmatrix}$

$\Rightarrow \cancel{2(0-5)}\ 2(0-5) - 0 + 3(1+4)$

$\Rightarrow -10 + 15$

$\Rightarrow 5$ Ans

Q6.) If $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$, find $|A|$

Sol. Expanding along R_3,

$5\begin{vmatrix} 1 & -2 \\ 1 & -3 \end{vmatrix} - 4\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} + (-9)\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$

$\Rightarrow 5(-3+2) - 4(-3+4) - 9(1-2)$

$\Rightarrow -5 - 4 + 9$

$\Rightarrow 0$ Ans

Q7.) Find values of x, if

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

(i) $2 - 20 = 2x^2 - 24$

$\Rightarrow 2x^2 - 24 + 18 = 0$

$\Rightarrow 2x^2 - 6 = 0$

$\Rightarrow x^2 - 3 = 0$

$\Rightarrow x^2 = 3$

$\Rightarrow x = \pm\sqrt{3}$ Ans

(ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

(ii) $10 - 12 = 5x - 6x$

$\Rightarrow x = 2$ Ans

Q8.) If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to

(A) 6 (B) ± 6 (C) -6 (D) 0

Sol. $x^2 - 36 = 36 - 36$

$\Rightarrow x^2 = 36$

$\Rightarrow x = \pm 6$

$\therefore$ (B) ± 6 Ans

23/4/23

★ EXERCISE - 4.2

Q1) Find area of the triangle with vertices at the point given in each of the following:

(i) $(1,0), (6,0), (4,3)$

(i) $$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

Expanding along C_2,

$$= \frac{1}{2}\left(-0\begin{vmatrix} 6 & 1 \\ 4 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 6 & 1 \end{vmatrix}\right)$$

$$= \frac{1}{2}\{0 + 0 - 3(1-6)\}$$

$$= \frac{15}{2} \text{ Ans}$$

(ii) $(2,7), (1,1), (10,8)$

(ii) $$\Delta = \frac{1}{2}\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$$

Expanding along R_1,

$$= \frac{1}{2}\left(2\begin{vmatrix} 1 & 1 \\ 8 & 1 \end{vmatrix} - 7\begin{vmatrix} 1 & 1 \\ 10 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 10 & 8 \end{vmatrix}\right)$$

$$= \frac{1}{2}[2(1-8) - 7(1-10) + 1(8-10)]$$

$$= \frac{1}{2}(-14 + 63 - 2)$$

$$= \frac{47}{2} \text{ Ans}$$

(iii) $(-2,-3), (3,2), (-1,-8)$

(iii) $\Delta = \frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ -3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}$

along C_2,

$= \frac{1}{2}\left[-(-3)\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} + 2\begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} - (-8)\begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix}\right]$

$= \frac{1}{2}\left[3(3+1) + 2(-2+1) + 8(-2-3)\right]$

$= \frac{1}{2}\left[12 - 2 - 40\right]$

$= \frac{30}{2}$

$= 15$ Ans

Q2.) Show that the points $A(a, b+c)$, $B(b, c+a)$, $C(c, a+b)$ are collinea

Sol. $\Delta = \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$

along C_1,

$= \frac{1}{2}\left[a\begin{vmatrix} c+a & 1 \\ a+b & 1 \end{vmatrix} - b\begin{vmatrix} a & 1 \\ c & 1 \end{vmatrix} + c\begin{vmatrix} b+c & 1 \\ c+a & 1 \end{vmatrix}\right]$

$= \frac{1}{2}\left[a(\cancel{b}+c-a-\cancel{b}) - b(a-c) + c(b+\cancel{c}-\cancel{c}-a)\right]$

$= ac - a^2 - ab + bc + bc$

Expanding along C_1,

$\Delta = \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$

$= \frac{1}{2}\left[a\begin{vmatrix} c+a & 1 \\ a+b & 1 \end{vmatrix} - b\begin{vmatrix} b+c & 1 \\ a+b & 1 \end{vmatrix} + c\begin{vmatrix} b+c & 1 \\ c+a & 1 \end{vmatrix}\right]$

$= \frac{1}{2}\left[a(c+\cancel{a}-\cancel{a}-b) - b(\cancel{b}+c-a-\cancel{b}) + c(b+\cancel{c}-\cancel{c}-a)\right]$

$= \dfrac{\cancel{ac} - \cancel{ab} - \cancel{bc} + \cancel{ab} + \cancel{bc} - \cancel{ac}}{2}$

$= 0$

$\Delta = 0$

$\therefore$ A, B and C are collinear

Q3.) Find values of k if area of triangle is 4 sq. units and vertices are

(i) $(k, 0), (4, 0), (0, 2)$

(i) $\Delta = \pm 4$

$\Rightarrow \dfrac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \pm 4$

Expanding along C_2,

$\Rightarrow \dfrac{1}{2}\left[-0\begin{vmatrix} 4 & 1 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} k & 1 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} k & 1 \\ 4 & 1 \end{vmatrix}\right] = \pm 4$

$\Rightarrow \dfrac{1}{2}\left[-0 + 0 - 2(k - 4)\right] = \pm 4$

$\Rightarrow \dfrac{-2k + 8}{2} = \pm 4$

$\Rightarrow -2k + 8 = \pm 8$

$+8$

$-2k + \cancel{8} = \cancel{8}$

$\Rightarrow k = 0$

-8

$-2k + 8 = -8$

$\Rightarrow -2k = -16$

$\Rightarrow k = 8$

$\boxed{k = 0, 8}$ Ans

(ii) $(-2, 0), (0, 4), (0, k)$

(ii) $\Delta = \pm 4$

$\Rightarrow \dfrac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix} = \pm 4$

Expansion along C_1

$\Rightarrow \frac{1}{2}\left[-2\begin{vmatrix}4 & 1\\ k & 1\end{vmatrix} - 0\begin{vmatrix}0 & 1\\ k & 1\end{vmatrix} + 0\begin{vmatrix}0 & 1\\ 4 & 1\end{vmatrix}\right] = \pm 4$

$\Rightarrow \frac{1}{2}\left[-2(4-k) - 0 + 0\right] = \pm 4$

$\Rightarrow \frac{-8+2k}{2} = \pm 4$

$\Rightarrow -8+2k = \pm 8$

$+8$ $\qquad\qquad$ -8

$-8+2k = 8$ $\qquad\qquad$ $-8+2k = -8$

$\Rightarrow 2k = 16$ $\qquad\qquad$ $\Rightarrow k = 0$

$\Rightarrow k = 8$

$\boxed{k = 0, 8}$ Ans

Q4.xi) Find equation of line joining $(1, 2)$ and $(3, 6)$ using determinants.

(i) $A(1, 2)$ $\qquad$ $B(3, 6)$

Let $P(x, y)$ be any point on AB

ABP are collinear

$\therefore \Delta = 0$

$\Rightarrow \Delta = \frac{1}{2}\begin{vmatrix}1 & 2 & 1\\ 3 & 6 & 1\\ x & y & 1\end{vmatrix}$

along C_1,

$\Rightarrow \frac{1}{2}\left[1\begin{vmatrix}6 & 1\\ y & 1\end{vmatrix} - 3\begin{vmatrix}2 & 1\\ y & 1\end{vmatrix} + x\begin{vmatrix}2 & 1\\ 6 & 1\end{vmatrix}\right] = 0$

$\Rightarrow \frac{1}{2}\left[6 - y - 3(2-y) + x(2-6)\right] = 0$

$\Rightarrow \frac{1}{2}\left[\cancel{6} - y - \cancel{6} + 3y + 2x - 6x\right] = 0$

$\Rightarrow \frac{2y - 4x}{2} = 0$

$\Rightarrow y = 2x$ Ans

(ii) Find equation of line joining (3, 1) and (9, 3) using determinants.

(ii) $A(3,1)$ $B(9,3)$

Let $P(x,y)$ be any point on AB

ABP are collinear

$\therefore \Delta = 0$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

along C_1,

$$\Rightarrow \frac{1}{2}\left[3\begin{vmatrix} 3 & 1 \\ y & 1 \end{vmatrix} - 9\begin{vmatrix} 1 & 1 \\ y & 1 \end{vmatrix} + x\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}\right] = 0$$

$$\Rightarrow \frac{1}{2}\left[3(3-y) - 9(1-y) + x(1-3)\right] = 0$$

$$\Rightarrow \frac{\cancel{9} - 3y - \cancel{9} + 9y + x - 3x}{2} = 0$$

$$\Rightarrow \frac{6y - 2x}{2} = 0$$

$$\Rightarrow 3y - x = 0$$

$$\Rightarrow x - 3y = 0$$ Ans

Q5.) If area of triangle is 35 sq units with vertices $(2,-6)$, $(5,4)$ and $(k,4)$. Then k is

(A) 12 (B) -2 (C) $-12, -2$ (D) $12, -2$

Sol. $\Delta = \pm 35$

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix} = \pm 35$$

Expanding along C_1,

$$\Rightarrow \frac{1}{2}\left[2\begin{vmatrix} 4 & 1 \\ 4 & 1 \end{vmatrix} - 5\begin{vmatrix} -6 & 1 \\ 4 & 1 \end{vmatrix} + k\begin{vmatrix} -6 & 1 \\ 4 & 1 \end{vmatrix}\right] = \pm 35$$

$$\Rightarrow \frac{1}{2}\left[2(4-4) - 5(-6-4) + k(-6-4)\right] = \pm 35$$

$$\Rightarrow \frac{30+20-6k-4k}{2} = \pm 35$$

$$\Rightarrow -10k + 50 = \pm 70$$

$$\Rightarrow -10k = \pm 70 - 50$$

$\underline{+70}$

$-10k = 70 - 50$

$\Rightarrow -10k = 20$

$\Rightarrow k = -2$

$\underline{-70}$

$-10k = -70 - 50$

$\Rightarrow -10k = -120$

$\Rightarrow k = 12$

$\therefore$ (b) 12, −2 Ans

23/4/21

★ EXERCISE - 4.3

- Write minors and cofactors of the elements of following determinants:

Q1.) (i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$

(i) $M_{11} = 3$; $A_{11} = (-1)^{1+1} M_{11} = 3$

$M_{12} = 0$; $A_{12} = (-1)^{1+2} M_{12} = 0$

$M_{21} = -4$; $A_{21} = (-1)^{2+1} M_{21} = 4$

$M_{22} = 2$; $A_{22} = (-1)^{2+2} M_{22} = 2$

(ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

(ii) $M_{11} = d$; $A_{11} = (-1)^{1+1} M_{11} = d$

$M_{12} = b$; $A_{12} = (-1)^{1+2} M_{12} = -b$

$M_{21} = c$; $A_{21} = (-1)^{2+1} M_{21} = -c$

$M_{22} = a$; $A_{22} = (-1)^{2+2} M_{22} = a$

Q2.) (i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

(i) $M_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$; $A_{11} = (-1)^{1+1} M_{11} = 1$

$M_{12} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 - 0 = 0$; $A_{12} = (-1)^{1+2} M_{12} = 0$

$M_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0$; $A_{13} = (-1)^{1+3} M_{13} = 0$

$M_{21} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 - 0 = 0$; $A_{21} = (-1)^{2+1} M_{21} = 0$

$M_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1-0 = 1$; $A_{22} = (-1)^{2+2} M_{22} = 1$

$M_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0-0 = 0$; $A_{23} = (-1)^{2+3} \cdot M_{23} = 0$

$M_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0-0 = 0$; $A_{31} = (-1)^{3+1} M_{31} = 0$

$M_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0-0 = 0$; $A_{32} = (-1)^{3+2} M_{32} = 0$

$M_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1-0 = 1$; $A_{33} = (-1)^{3+3} M_{33} = 1$

(ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

(ii) $M_{11} = \begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10+1 = 11$; $A_{11} = (-1)^{1+1} M_{11} = 11$

$M_{12} = \begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6-0 = 6$; $A_{12} = (-1)^{1+2} M_{12} = -6$

$M_{13} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3-0 = 3$; $A_{13} = (-1)^{1+3} \cdot M_{13} = 3$

$M_{21} = \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 0-4 = -4$; $A_{21} = (-1)^{2+1} M_{13} = 4$

$M_{22} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2-0 = 2$; $A_{22} = (-1)^{2+2} M_{22} = 2$

$M_{23} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1-0 = 1$; $A_{23} = (-1)^{2+3} M_{23} = -1$

$M_{31} = \begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = 0-20 = -20$; $A_{31} = (-1)^{3+1} M_{31} = -20$

$M_{32} = \begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1-12 = -13$; $A_{32} = (-1)^{3+2} M_{32} = 13$

$M_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5 - 0 = 5$; $A_{33} = (-1)^{3+3} M_{33} = 5$

Q3.) Using Cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

SoL. $M_{21} = \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = 9 - 16 = -7$; $A_{21} = (-1)^{2+1} M_{21} = 7$

$M_{22} = \begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = 15 - 8 = 7$; $A_{22} = (-1)^{2+2} \cdot M_{22} = 7$

$M_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7$; $A_{23} = (-1)^{2+3} M_{23} = -7$

$\Delta = a_{21} A_{21} + a_{22} A_{22} \; a_{23} A_{23}$

$\Rightarrow \Delta = 2 \times 7 + 0 \times 7 + 1 \times -7$

$\Rightarrow \Delta = 14 + 0 - 7$

$\Rightarrow \Delta = 7$ Ans

Q4.) Using Cofactors of elements of third column, evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$

SoL. $M_{13} = \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = z - y$; $A_{13} = (-1)^{1+3} M_{13} = z - y$

$M_{23} = \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = z - x$; $A_{23} = (-1)^{2+3} \cdot M_{23} = -(z - x) = x - z$

$M_{33} = \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = y - x$; $A_{33} = (-1)^{3+3} M_{33} = y - x$

$\Delta = yz(z-y) + zx(x-z) + xy(y-x)$

$\Rightarrow \Delta = yz(z-y) + zx^2 - z^2x + xy^2 - x^2y$

$\Rightarrow \Delta = yz(z-y) + x^2(z-y) - x(z^2-y^2)$

$\Rightarrow \Delta = (z-y)(yz + x^2 - x(z+y))$

$\Rightarrow \Delta = (z-y)(yz + x^2 - xz - xy)$

$\Rightarrow \Delta = (z-y)(x(x-y) - z(x-y))$

$\Rightarrow \Delta = (x-y)(z-y)(x-z)$

$\Rightarrow \Delta = (x-y)(-1)(y-z)(-1)(z-x)$

$\Rightarrow \Delta = (x-y)(y-z)(z-x)$ Ans

Q5.) If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is Cofactors of a_{ij}, then value of Δ is given by

(A) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$

(B) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$

(C) $a_{21}a_{11} + a_{22}A_{12} + a_{23}A_{13}$

(D) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

25/4/23

★ $A(\text{adj } A) = |A| I = (\text{adj} A) A$ $\qquad |A| = |A^t|$

$|(\text{adj} A)| = |A|^{n-1}$ $\qquad (A^t)^{-1} = (A^{-1})^t$

A is a square matrix of order n.

★ INVERSE OF A MATRIX

A square matrix A is invertible if and only if A is non-singular matrix.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \qquad |A| \neq 0$$

$A_{11} = +\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 16 - 9 = 7$

$A_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -(4-3) = -1$

$A_{13} = \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 3 - 4 = -1$

$A_{21} = -\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = -(12-9) = -3$

$A_{22} = +\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$

$A_{23} = -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$

$A_{31} = \begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} = 9 - 12 = -3$

$A_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$

$A_{33} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1$

$$\text{adj} A = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^t = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$|A| = 1(16-9) - 3(4-3) + \cancel{3(4-3)}\ 3(3-4)$

$= 7 - 3 - 3$

$= 1 \neq 0$

A is invertible

* $\boxed{A^{-1} = \frac{1}{|A|} (\text{adj } A)}$

$$= \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

25/4/23

☆ EXERCISE – 4.4

- Find adjoint of each of the matrices in Exercises 1 and 2.

Q1.) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Ans $A_{11} = 4$; $A_{12} = -3$

$A_{21} = -2$; $A_{22} = 1$

$\cancel{|A| = 1(4) - 2(-3)}$

$\cancel{= 4 - 6}$

$\cancel{= -2}$

$\text{adj } A = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^t = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$ Ans

---

Q2.) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

Ans $A_{11} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3$ $A_{12} = -\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -12$ $A_{13} = \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = +6$

$A_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = 1$ $A_{22} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 5$ $A_{23} = -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = +2$

$A_{31} = \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -11$ $A_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -1$ $A_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5$

$\text{adj } A = \begin{bmatrix} 3 & 8 & +6 \\ 1 & 5 & 2 \\ -11 & -1 & 5 \end{bmatrix}^t = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ +6 & 2 & 5 \end{bmatrix}$ Ans

- Verify $A(\text{adj } A) = (\text{adj } A)A = |A| I$ in Exercises 3 and 4

Q3.) $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$

Ans $A_{11} = -6 \qquad A_{12} = 4$

$A_{21} = -3 \qquad A_{22} = 2$

$\text{adj } A = \begin{bmatrix} -6 & 4 \\ -3 & 2 \end{bmatrix}^t = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$

$|A| = 2(-6) + 3(4)$

$= -12 + 12$

$= 0$

$A(\text{adj } A) = (\text{adj } A)A = |A| I$

$\Rightarrow \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix} = 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -12+12 & -6+6 \\ 24-24 & 12-12 \end{bmatrix} = \begin{bmatrix} -12+12 & -18+18 \\ 8-8 & 12-12 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Hence, verified.

Q4.) $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

Ans $A_{11} = \begin{vmatrix} 0 & -2 \\ 0 & 3 \end{vmatrix} \qquad A_{12} = -\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix} \qquad A_{13} = \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix}$

$= 0 \qquad = -11 \qquad = 0$

$A_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 3 \end{vmatrix} \qquad A_{22} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \qquad A_{23} = -\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}$

$= 3 \qquad = 1 \qquad = -1$

$A_{31} = \begin{vmatrix} -1 & 2 \\ 0 & -2 \end{vmatrix} = 2$ $\quad A_{32} = -\begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = 8$ $\quad A_{33} = \begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} = 3$

$|A| = 1\times 0 + (-1)(-11) + 2\times 0 = 11$

~~$A(\text{adj } A) = (\text{adj } A)A = |A|I$~~

$$\text{adj} A = \begin{bmatrix} 0 & -11 & 0 \\ 3 & 1 & -1 \\ 2 & 8 & 3 \end{bmatrix}^t = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

$A(\text{adj } A) = (\text{adj } A)A = |A|I$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix} = 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0+11+0 & 3-1-2 & 2-8+6 \\ 0+0+0 & 9+0+2 & 6+0-6 \\ 0+0+0 & 3+0-3 & 2+0+9 \end{bmatrix} = \begin{bmatrix} 0+9+2 & 0+0+0 & 0+(-6)+6 \\ -11+3+8 & 11+0+0 & -22-2+24 \\ 0-3+3 & 0+0+0 & 0+2+9 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

Hence, verified

- Find the inverse of each of the matrices (if it exists) given in Exercises 5 to 11.

Q5.) $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

Sol. $A_{11} = 3 \qquad A_{12} = -4$

$A_{21} = +2 \qquad A_{22} = 2$

$|A| \Rightarrow 2\times 3 + (-2)(-4) \Rightarrow 6+8 \Rightarrow 14$ (Inverse exists)

$$\text{adj } A = \begin{bmatrix} 3 & -4 \\ +2 & 2 \end{bmatrix}^t = \begin{bmatrix} 3 & +2 \\ -4 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj} A) = \frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$$ Ans

---

Q6.) $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$

Sol. $A_{11} = 2$ $\quad A_{12} = 3$

$A_{21} = -5$ $\quad A_{22} = -1$

$|A| \Rightarrow -1(2) + 5(3) \Rightarrow -2+15 \Rightarrow 13$ (inverse exists)

$$\text{adj } A = \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}^t = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$$ Ans

---

Q7.) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

Sol. $A_{11} = \begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} = 10$ $\quad A_{12} = -\begin{vmatrix} 0 & 4 \\ 0 & 5 \end{vmatrix} = 0$ $\quad A_{13} = \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0$

$A_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = -10$ $\quad A_{22} = \begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = 5$ $\quad A_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0$

$A_{31} = \begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} = 2$ $\quad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = -4$ $\quad A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2$

$|A| = 10$ (inverse exists)

$$\text{adj } A = \begin{bmatrix} 10 & 0 & 0 \\ -10 & 5 & 0 \\ 2 & -4 & 2 \end{bmatrix}^t = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix} \text{ Ans}$$

Q8.) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

Sol. $A_{11} = \begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} = -3$ $\quad A_{12} = -\begin{vmatrix} 3 & 0 \\ 5 & -1 \end{vmatrix} = 3$ $\quad A_{13} = \begin{vmatrix} 3 & 3 \\ 5 & 2 \end{vmatrix} = -9$

$A_{21} = -\begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} = 0$ $\quad A_{22} = \begin{vmatrix} 1 & 0 \\ 5 & -1 \end{vmatrix} = -1$ $\quad A_{23} = -\begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} = -2$

$A_{31} = \begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} = 0$ $\quad A_{32} = -\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = 0$ $\quad A_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 3 \end{vmatrix} = 3$

$|A| \Rightarrow 1(-3) + 0 + 0 \Rightarrow -3$ (inverse exists)

$$\text{adj } A = \begin{bmatrix} -3 & 3 & -9 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{bmatrix}^t = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj} A) = \frac{1}{-3}\begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

Q9.) $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$

Sol. $A_{11} = \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = -1$ $\quad A_{12} = -\begin{vmatrix} 4 & 0 \\ -7 & 1 \end{vmatrix} = -4$ $\quad A_{13} = \begin{vmatrix} 4 & -1 \\ -7 & 2 \end{vmatrix} = 1$

$A_{21} = -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 5$ $\quad A_{22} = \begin{vmatrix} 2 & 3 \\ -7 & 1 \end{vmatrix} = 23$ $\quad A_{23} = -\begin{vmatrix} 2 & 1 \\ -7 & 2 \end{vmatrix} = -11$

$A_{31} = \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} = 3$ $\quad A_{32} = -\begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} = 12$ $\quad A_{33} = \begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} = -6$

$|A| \Rightarrow 2(-1) + 1(-4) + 3(1) \Rightarrow -2 - 4 + 3 \Rightarrow -3$ (inverse exists)

$\text{adj } A = \begin{bmatrix} -1 & -4 & 1 \\ 5 & 23 & -11 \\ 3 & 12 & -6 \end{bmatrix}^t = \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} (\text{adj } A) = \frac{1}{-3} \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$ Ans

---

Q10.) $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

Sol. $A_{11} = \begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} = 2$ $\quad A_{12} = -\begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix} = -9$ $\quad A_{13} = \begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} = -6$

$A_{21} = -\begin{vmatrix} -1 & 2 \\ -2 & 4 \end{vmatrix} = 0$ $\quad A_{22} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$ $\quad A_{23} = -\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix} = -1$

$A_{31} = \begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix} = -1$ $\quad A_{32} = -\begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix} = 3$ $\quad A_{33} = \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2$

$|A| \Rightarrow 1(2) + (-1)(-9) + 2(-6) \Rightarrow 2 + 9 - 12 \Rightarrow -1$ (inverse exists)

$$\text{adj } A = \begin{bmatrix} 2 & -9 & -6 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{bmatrix}^t = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj} A) = \frac{1}{-1}\begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix} \quad \underline{\text{Ans}}$$

Q11.) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

Sol. $A_{11} = \begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{vmatrix}$
$= -\cos^2\alpha - \sin^2\alpha$
$= -(\sin^2\alpha + \cos^2\alpha)$
$= -1$

$A_{12} = -\begin{vmatrix} 0 & \sin\alpha \\ 0 & -\cos\alpha \end{vmatrix} = 0$

$A_{13} = \begin{vmatrix} 0 & \cos\alpha \\ 0 & \sin\alpha \end{vmatrix} = 0$

$A_{21} = -\begin{vmatrix} 0 & 0 \\ \sin\alpha & -\cos\alpha \end{vmatrix} = 0$

$A_{22} = \begin{vmatrix} 1 & 0 \\ 0 & -\cos\alpha \end{vmatrix} = -\cos\alpha$

$A_{23} = -\begin{vmatrix} 1 & 0 \\ 0 & \sin\alpha \end{vmatrix} = -\sin\alpha$

$A_{31} = \begin{vmatrix} 0 & 0 \\ \cos\alpha & \sin\alpha \end{vmatrix} = 0$

$A_{32} = -\begin{vmatrix} 1 & 0 \\ 0 & \sin\alpha \end{vmatrix} = -\sin\alpha$

$A_{33} = \begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} = \cos\alpha$

$|A| = 1(-1) + 0 + 0 \Rightarrow -1$ (inverse exists)

$$\text{adj } A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}^t = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{-1}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$$

Q12.) Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Sol. $AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 18+49 & 24+63 \\ 12+35 & 16+45 \end{bmatrix} = \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$

$$\begin{array}{r} 67 \\ \times 61 \\ \hline 67 \\ 402\times \\ \hline 4087 \end{array} \qquad \begin{array}{r} 87 \\ \times 47 \\ \hline 609 \\ 348\times \\ \hline 4089 \end{array}$$

$$\boxed{\cancel{\begin{aligned} |AB| &= (67\times 61) - (87\times 47) \\ &= 4087 - 4089 \\ &= -2 \quad (\text{inverse exists}) \end{aligned}}}$$

$(AB)_{11} = 61$ $\quad (AB)_{12} = -47$

$(AB)_{21} = -87$ $\quad (AB)_{22} = 67$

$|AB| \Rightarrow 67\times 61 + 87(-47) \Rightarrow 4087 - 4089 \Rightarrow -2$ (inverse exists)

$\text{adj}(AB) = \begin{bmatrix} 61 & -47 \\ -87 & 67 \end{bmatrix}^t = \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$

$(AB)^{-1} = \frac{1}{|A|}[\text{adj}(AB)] = \frac{1}{-2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$

L.H.S

$B_{11} = 9$ $\quad B_{12} = -7$

$B_{21} = -8$ $\quad B_{22} = 6$

$|B| \Rightarrow 6\times 9 + 8(-7) \Rightarrow 54 - 56 \Rightarrow -2$ (inverse exists)

$\text{adj } B = \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}^t = \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$

$B^{-1} = \frac{1}{|B|}(\text{adj } B) = \frac{1}{-2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$

$A_{11} = 5 \qquad A_{12} = -2$

$A_{21} = -7 \qquad A_{22} = 3$

$|A| \Rightarrow 3\times5 + 7(-2) \Rightarrow 15-14 = 1$

$adj A = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}^t = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}(adj A) = \frac{1}{1}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

$B^{-1}A^{-1} = \frac{1}{-2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 45+16 & -63-24 \\ -35-12 & 49+18 \end{bmatrix}$

$\Rightarrow B^{-1}A^{-1} = \frac{1}{-2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$

R.H.S

L.H.S. = R.H.S.

Hence, verified

Q13.) If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$. Hence find A^{-1}.

Sol. $A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

$A^2 - 5A + 7I$

$\Rightarrow \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Hence, proved.

$A^2 - 5A + 7I = 0$

Pre-multiplying by A^{-1};

$\Rightarrow A^{-1}A^2 - 5A^{-1}A + 7A^{-1}I = A^{-1}O$

$\Rightarrow (A^{-1}A)A - 5I + 7A^{-1} = O$

$\Rightarrow IA - 5I + 7A^{-1} = O$

$\Rightarrow A - 5I + 7A^{-1} = O$

$\Rightarrow 7A^{-1} = -A + 5I$

$\Rightarrow 7A^{-1} = -\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow 7A^{-1} = \begin{bmatrix} -3 & -1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$\Rightarrow 7A^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$\Rightarrow A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ Ans

---

Q14.) For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = O$.

Sol. $A^2 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9+2 & 6+2 \\ 3+1 & 2+1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$

$A^2 + aA + bI = O$

$\Rightarrow \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + a\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 3a & 2a \\ a & a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 11+3a+b & 8+2a \\ 4+a & 3+a+b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$11 + 3a + b = 0$ — ①

$8 + 2a = 0$ — ②

$$4 + a = 0 \quad \text{---} ③$$
$$3 + a + b = 0 \quad \text{---} ④$$

① − ④

$$11 + 3a + \cancel{b} = 0$$
$$3 + a + \cancel{b} = 0$$
$$- \quad - \quad -$$
$$8 + 2a = 0$$
$$2a = -8$$
$$\Rightarrow a = -4$$

$$④ \Rightarrow 3 - 4 + b = 0$$
$$\Rightarrow b = 1$$

$a = -4,\ b = 1$ Ans

Q15.) For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ Show that $A^3 - 6A^2 + 5A + 11I = 0$. Hence, find A^{-1}.

Sol. $$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1+1+2 & 1+2-1 & 1+(-3)+3 \\ 1+2-6 & 1+4+3 & 1+(-6)-9 \\ 2+(-1)+6 & 2-2-3 & 2+3+9 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$A^3 = A.A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} = \begin{bmatrix} 4-3+7 & 2+8-3 & 1-14+14 \\ 4-6-21 & 2+16+9 & 1-28-42 \\ 8+3+21 & 4-8-9 & 2+14+42 \end{bmatrix}$$

$$\Rightarrow A^3 = \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

$A^3 - 6A^2 + 5A + 11I$

$$\Rightarrow \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8-24+5+11 & 7-12+5+0 & 1-6+5+0 \\ -23+18+5+0 & 27-48+10+11 & -69+84-15+0 \\ 32-42+10+0 & -13+18-5+0 & 58-84+15+11 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow 0$

Hence, verified

$A^3 - 6A^2 + 5A + 11I = 0$

Pre-multiplying by A^{-1},

$\Rightarrow A^{-1}A^3 - 6A^{-1}A^2 + 5A^{-1}A + 11A^{-1}I = A^{-1}0$

$\Rightarrow (A^{-1}A)A^2 - 6(A^{-1}A)A + 5I + 11A^{-1} = 0$

$\Rightarrow IA^2 - 6IA + 5I + 11A^{-1} = 0$

$\Rightarrow A^2 - 6A + 5I + 11A^{-1} = 0$

$\Rightarrow 11A^{-1} = -A^2 + 6A - 5I$

$$\Rightarrow 11A^{-1} = -\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow 11A^{-1} = \begin{bmatrix} -4 & -2 & -1 \\ 3 & -8 & 14 \\ -7 & 3 & -14 \end{bmatrix} + \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\Rightarrow 11A^{-1} = \begin{bmatrix} -4+6-5 & -2+6-0 & -1+6-0 \\ 3+6-0 & -8+12-5 & 14-18-0 \\ -7+12-0 & 3-6-0 & -14+18-5 \end{bmatrix}$$

$$\Rightarrow 11A^{-1} = \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix} \quad \text{Ans}$$

Q16.) If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ Verify that $A^3 - 6A^2 + 9A - 4I = 0$ and hence find A^{-1}.

Sol. $$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A.A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} = \begin{bmatrix} 12+5+5 & -10-6-5 & 10+5+6 \\ -6-10-5 & 5+12+5 & -5-10-6 \\ 6+5+10 & -5-6-10 & 5+5+12 \end{bmatrix}$$

$$\Rightarrow A^3 = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$A^3 - 6A^2 + 9A - 4I$

$$\Rightarrow \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 22-36+18-4 & -21+30-9-0 & 21-30+9-0 \\ -21+30-9-0 & 22-36+18-4 & -21+30-9-0 \\ 21-30+9-0 & -21+30-9-0 & 22-36+18-4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow 0$

Hence, verified

$A^3 - 6A^2 + 9A - 4I = 0$

Pre-multiplying by A^{-1},

$\Rightarrow A^{-1}A^3 - 6A^{-1}A^2 + 9A^{-1}A - 4A^{-1}I = A^{-1}0$

$\Rightarrow (A^{-1}A)A^2 - 6(A^{-1}A)A + 9I - 4A^{-1} = 0$

$\Rightarrow IA^2 - 6IA + 9I - 4A^{-1} = 0$

$\Rightarrow A^2 - 6A + 9I - 4A^{-1} = 0$

$\Rightarrow 4A^{-1} = A^2 - 6A + 9I$

$$\Rightarrow 4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow 4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\Rightarrow 4A^{-1} = \begin{bmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{bmatrix}$$

$$\Rightarrow 4A^{-1} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$ Ans

Q17.) Let A be a non-singular square matrix of order 3×3. Then $|adj\ A|$ is equal to.

(A) $|A|$ (B) $|A|^2$ (C) $|A|^3$ (D) $3|A|$

Sol. If A is a square matrix of order n,

$$|adj\ (A)| = |A|^{n-1}$$

Given order $\Rightarrow 3 \times 3$

$$|adj\ A| = |A|^{3-1} = |A|^2$$

$\therefore$ (B) $|A|^2$ Ans

Q18.) If A is an invertible matrix of order 2, then $\det(A^{-1})$ is equal to

(A) $\det(A)$ (B) $\dfrac{1}{\det(A)}$ (C) 1 (D) 0

Sol. $|A| \neq 0$

$$A A^{-1} = I$$

$\Rightarrow |A|\,|A^{-1}| = |I|$

$\Rightarrow |A|\,|A^{-1}| = 1$

$\Rightarrow |A^{-1}| = \dfrac{1}{|A|}$

$\Rightarrow \det(A^{-1}) = \dfrac{1}{\det(A)}$

$\therefore$ (B) $\dfrac{1}{\det(A)}$ Ans

26/4/23

★ SOLUTION OF LINEAR EQUATIONS

* CONSISTENT — Unique solution or infinite solutions
* INCONSISTENT — No solution

$$\left.\begin{aligned} a_1x + b_1y &= 0 \\ a_2x + b_2y &= 0 \end{aligned}\right\} \text{Homogeneous}$$

$$\left.\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}\right\} \text{Non-homogeneous}$$

* For non-singular matrix ($|A| \neq 0$), solution is unique

$3x - 2y + 3 = 8$
$2x + y - z = 1$
$4x - 3y + 2z = 4$

$$\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$AX = B$
$\Rightarrow (A^{-1}A)X = A^{-1}B$
$\Rightarrow IX = A^{-1}B$
$\Rightarrow X = A^{-1}B$

$|A| = 3(2-3) + 2(4+4) + 3(-6-4)$
$= 3(-1) + 2(8) + 3(-10)$
$= -3 + 16 - 30$
$= -17 \neq 0$ (inverse exists)

$A_{11} = \cancel{4} \; 2-3 = -1$ $\quad A_{12} = -(4+4) = -8$ $\quad A_{13} = -6-4 = -10$

$A_{21} = -(-4+9) = -5$ $\quad A_{22} = 6-12 = -6$ $\quad A_{23} = -(-9+8) = 1$

$A_{31} = 2-3 = -1$ $\quad A_{32} = -(-3-6) = 9$ $\quad A_{33} = 3+4 = 7$

$$\text{adj} A = \begin{bmatrix} -1 & -8 & -10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{bmatrix}^t = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \cdot (\text{adj} A) = \frac{1}{-17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 1 & 5 & 1 \\ 8 & 6 & -9 \\ 10 & -1 & -7 \end{bmatrix}$$

$$X = \cancel{AB} \; A^{-1} B = \frac{1}{17} \begin{bmatrix} 1 & 5 & 1 \\ 8 & 6 & -9 \\ 10 & -1 & -7 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 1 \times 8 + 5 \times 1 + 1 \times 4 \\ 64 + 6 - 36 \\ 80 - 1 - 28 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{17} \begin{bmatrix} 17 \\ 34 \\ 51 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$x = 1, \; y = 2, \; z = 3$

26/4/21

☆ EXERCISE – 4.5

- Examine the consistency of the system of equations in Exercises 1 to 6.

Q1.) $x + 2y = 2$

$2x + 3y = 3$

Sol.

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$AX = B$$

$$\Rightarrow (A^{-1}A)X = A^{-1}B$$

$$\Rightarrow IX = A^{-1}B$$

$$\Rightarrow X = A^{-1}B$$

$|A| \Rightarrow 3 - 4 \Rightarrow -1$ ~~(inverse exists)~~ $\neq 0$

A is non-singular matrix

$\therefore$ The system of equations is consistent.

Q2.) $2x - y = 5$

$x + y = 4$

Sol. $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

$|A| = 2 + 1 = 3 \neq 0$

A is non-singular matrix

$\therefore$ The system of equations is consistent.

Q3.) $x + 3y = 5$

$2x + 6y = 8$

Sol. $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

$|A| = 6 - 6 = 0$

A is singular matrix

$\therefore$ The system of equations is inconsistent.

Q4.) $x + y + z = 1$

$2x + 3y + 2z = 2$

$ax + ay + 2az = 4$

Sol. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ a & a & 2a \end{bmatrix}$

$|A| = 1(6a - 2a) + 1(4a - 2a) + 1(2a - 3a)$

$= 4a + 2a - a$

$= 5a \neq 0$

A is non-singular matrix

$\therefore$ The system of equations is consistent

Q5.) $3x - y - 2z = 2$

$2y - z = -1$

$3x - 5y = 3$

Sol. $A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}$

$|A| = 3(-5) + 0 + 3(5)$

$= -15 + 15$

$= 0$

A is singular matrix

$\therefore$ The system of equations is inconsistent.

Q6.) $5x - y + 4z = 5$

$2x + 3y + 5z = 2$

$5x - 2y + 6z = -1$

Sol. $A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}$

$|A| = 5(18+10) - 1(12-25) + 4(-4-15)$

$= 140 + 13 - \cancel{78}76$

$= 77 \neq 0$

$153 - 76 = 77$

A is non-singular matrix

$\therefore$ The system of equations is consistent

- Solve system of linear equations, using matrix method, in Exercises 7 to 14.

Q7.) $5x + 2y = 4$

$7x + 3y = 5$

Sol. $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$AX = B$

$\Rightarrow (A^{-1}A)X = A^{-1}B$

$\Rightarrow IX = A^{-1}B$

$\Rightarrow X = A^{-1}B$

$A_{11} = 3 \qquad A_{12} = -7$

$A_{21} = -2 \qquad A_{22} = 5$

$|A| \Rightarrow 3 \times 5 + (-7)(2) \Rightarrow 15 - 14 \Rightarrow 1$ (inverse exists)

$\text{adj } A = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}^t = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 12-10 \\ -28+25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$x = 2$, $y = -3$ Ans

Q8.) $2x - y = -2$

$3x + 4y = 3$

Sol. $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$$AX = B$$
$$\Rightarrow (A^{-1}A)X = A^{-1}B$$
$$\Rightarrow IX = A^{-1}B$$
$$\Rightarrow X = A^{-1}B$$

$A_{11} = 4$ $\qquad A_{12} = -3$

$A_{21} = 1$ $\qquad A_{22} = 2$

$|A| \Rightarrow 4(2) + (-3)(-1) \Rightarrow 8 + 3 \Rightarrow 11$ (inverse exists)

$$\text{adj } A = \begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}^t = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -8+3 \\ 6+6 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -5 \\ 12 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5/11 \\ 12/11 \end{bmatrix}$

$x = \frac{-5}{11}$, $y = \frac{12}{11}$ Ans

Q9.) $4x - 3y = 3$
$3x - 5y = 7$

Sol. $\begin{bmatrix} 4 & -3 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$AX = B$
$\Rightarrow (A^{-1}A)X = A^{-1}B$
$\Rightarrow IX = A^{-1}B$
$\Rightarrow X = A^{-1}B$

$A_{11} = -5 \qquad A_{12} = -3$
$A_{21} = 3 \qquad A_{22} = 4$

$|A| \Rightarrow -5(4) + (-3)(-3) \Rightarrow -20 + 9 \Rightarrow -11$ (inverse exists)

$\text{adj } A = \begin{bmatrix} -5 & -3 \\ 3 & 4 \end{bmatrix}^t = \begin{bmatrix} -5 & 3 \\ -3 & 4 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{-11} \begin{bmatrix} -5 & 3 \\ -3 & 4 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 5 & -3 \\ 3 & -4 \end{bmatrix}$

$X = A^{-1}B = \frac{1}{11} \begin{bmatrix} 5 & -3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 15-21 \\ 9-28 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -6 \\ -19 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6/11 \\ -19/11 \end{bmatrix}$

$x = \frac{-6}{11}$, $y = \frac{-19}{11}$ Ans

Q10.) $5x+2y=3$

$3x+2y=5$

Sol. $\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$AX = B$

$\Rightarrow (A^{-1}A)X = A^{-1}B$

$\Rightarrow IX = A^{-1}B$

$\Rightarrow X = A^{-1}B$

$A_{11} = 2 \qquad A_{12} = -3$

$A_{21} = -2 \qquad A_{22} = 5$

$|A| \Rightarrow 2(5) + (-3)(2) \Rightarrow 10-6 \Rightarrow 4$ (inverse exists)

$\text{adj } A = \begin{bmatrix} 2 & -3 \\ -2 & 5 \end{bmatrix}^t = \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$

$X = A^{-1}B = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6-10 \\ -9+25 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -4 \\ 16 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

$x=-1,\ y=4$ Ans

Q11.) $2x+y+z=1$

$x-2y-z=\frac{3}{2}$

$3y-5z=9$

Sol. $\begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ 9 \end{bmatrix}$

$$AX = B$$
$$\Rightarrow (A^{-1}A)X = A^{-1}B$$
$$\Rightarrow IX = A^{-1}B$$
$$\Rightarrow X = A^{-1}B$$

$A_{11} = 10+3 = 13$ $\quad$ $A_{12} = 5$ $\quad$ $A_{13} = 3$

$A_{21} = 8$ $\quad$ $A_{22} = -10$ $\quad$ $A_{23} = -6$

$A_{31} = 1$ $\quad$ $A_{32} = 3$ $\quad$ $A_{33} = -5$

$|A| \Rightarrow 13(2) + 5(1) + 3(1) \Rightarrow 26+5+3 \Rightarrow 34$ (inverse exists)

$$\text{adj } A = \begin{bmatrix} 13 & 5 & 3 \\ 8 & -10 & -6 \\ 1 & 3 & -5 \end{bmatrix}^t = \begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{34}\begin{bmatrix} 13 & 8 & 1 \\ 5 & -10 & 3 \\ 3 & -6 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 3/2 \\ 9 \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 13+12+9 \\ 5-15+27 \\ 3-9-45 \end{bmatrix} = \frac{1}{34}\begin{bmatrix} 34 \\ 17 \\ -51 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ -3/2 \end{bmatrix}$$

$x = 1,\ y = \frac{1}{2},\ z = \frac{-3}{2}$ Ans

Q12) $x - y + z = 4$

$2x + y - 3z = 0$

$x + y + z = 2$

Sol.

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$AX = B$$
$$\Rightarrow (A^{-1}A)X = A^{-1}B$$
$$\Rightarrow IX = A^{-1}B$$
$$\Rightarrow X = A^{-1}B$$

$A_{11} = 4$ $\quad A_{12} = -5$ $\quad A_{13} = 1$

$A_{21} = 2$ $\quad A_{22} = 0$ $\quad A_{23} = -2$

$A_{31} = 2$ $\quad A_{32} = 5$ $\quad A_{33} = 3$

$|A| \Rightarrow 4(1) + (-5)(-1) + (1)(1) \Rightarrow 4 + 5 + 1 \Rightarrow 10$ (inverse exists)

$$\text{adj } A = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}^t = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{10} \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{10} \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4+0+6 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$x = 2,\ y = -1,\ z = 1$ Ans

Q13.) $2x + 3y + 3z = 5$

$x - 2y + z = -4$

$3x - y - 2z = 3$

Sol. $\begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$

$AX = B$

$\Rightarrow (A^{-1}A)X = A^{-1}B$

$\Rightarrow IX = A^{-1}B$

$\Rightarrow X = A^{-1}B$

$A_{11} = 5$ $\quad A_{12} = 5$ $\quad A_{13} = 5$

$A_{21} = 3$ $\quad A_{22} = -13$ $\quad A_{23} = 11$

$A_{31} = 9$ $\quad A_{32} = 1$ $\quad A_{33} = -7$

$|A| \Rightarrow 5(2) + 5(3) + 5(3) \Rightarrow 10 + 15 + 15 \Rightarrow 40$ (inverse exists)

$\text{adj } A = \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}^t = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$

$X = A^{-1}B = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 25 - 12 + 27 \\ 25 + 52 + 3 \\ 25 - 44 - 21 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$x = 1, y = 2, z = -1$ Ans

Q14.) $x - y + 2z = 7$

$3x + 4y - 5z = -5$

$2x - y + 3z = 12$

Sol. $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$

$$AX = B$$

$$\Rightarrow (A^{-1}A)X = A^{-1}B$$

$$\Rightarrow IX = A^{-1}B$$

$$\Rightarrow X = A^{-1}B$$

$A_{11} = 7$ $\quad A_{12} = -19$ $\quad A_{13} = -11$

$A_{21} = 1$ $\quad A_{22} = -1$ $\quad A_{23} = -1$

$A_{31} = -3$ $\quad A_{32} = 11$ $\quad A_{33} = 7$

$|A| \Rightarrow 7(1) + (-19)(-1) + (-11)(2) \Rightarrow 7 + 19 - 22 \Rightarrow 4$ (inverse exists)

$$\text{adj } A = \begin{bmatrix} 7 & -19 & -11 \\ 1 & -1 & -1 \\ -3 & 11 & 7 \end{bmatrix}^t = \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 49 - 5 - 36 \\ -133 + 5 + 132 \\ -77 + 5 + 84 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$x = 2,\ y = 1,\ z = 3$ Ans

Q15.) If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Using A^{-1} solve the system of equations

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

Sol. $A_{11} = 0$ $\quad A_{12} = 2$ $\quad A_{13} = 1$

$A_{21} = -1$ $\quad A_{22} = -9$ $\quad A_{23} = -5$

$A_{31} = 2$ $\quad A_{32} = 23$ $\quad A_{33} = 13$

$|A| \Rightarrow 0(2) + 2(-3) + 1(5) \Rightarrow 0 - 6 + 5 \Rightarrow -1$ (inverse exists)

$$\text{adj } A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^t = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = -\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$AX = B$$
$$\Rightarrow (A^{-1}A)X = A^{-1}B$$
$$\Rightarrow IX = A^{-1}B$$
$$\Rightarrow X = A^{-1}B$$

$$X = A^{-1}B = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$x = 1, y = 2, z = 3$ Ans

Q16.) The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹90. The cost of 6 kg onion, 2 kg wheat and 3 kg rice is ₹70. Find cost of each item per kg by matrix method.

Sol. Let cost of 1 kg onion be ₹ x

Let cost of 1 kg wheat be ₹ y

Let cost of 1 kg rice be ₹ z

A.T.Q.

$$4x + 3y + 2z = 60$$
$$2x + 4y + 6z = 90$$
$$6x + 2y + 3z = 70$$

$$\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$AX = B$$
$$\Rightarrow (A^{-1}A)X = A^{-1}B$$
$$\Rightarrow IX = A^{-1}B$$
$$\Rightarrow X = A^{-1}B$$

$A_{11} = 0$ $\quad A_{12} = 30$ $\quad A_{13} = -20$

$A_{21} = -5$ $\quad A_{22} = 0$ $\quad A_{23} = 10$

$A_{31} = 10$ $\quad A_{32} = -20$ $\quad A_{33} = 10$

$|A| \Rightarrow 0(4) + 30(3) + (-20)(2) \Rightarrow 0 + 90 - 40 \Rightarrow 50$ (inverse exists)

$$\text{adj } A = \begin{bmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{bmatrix}^t = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 0-450+700 \\ 1800+0-1400 \\ -1200+900+700 \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$x = 5$, $y = 8$, $z = 8$

Price of onion = ₹5/kg
Price of wheat = ₹8/kg
Price of rice = ₹8/kg

EXTRA QUESTION

Q. $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ Find A^{-1} and solve

$x + 2y + z = 4$
$-x + y + z = 0$
$x - 3y + z = 2$

Sol. $A_{11} = 4$ $\quad A_{12} = -5$ $\quad A_{13} = 1$
$A_{21} = 2$ $\quad A_{22} = 0$ $\quad A_{23} = -2$
$A_{31} = 2$ $\quad A_{32} = 5$ $\quad A_{33} = 3$

$|A| \Rightarrow 4(1) + (-5)(-1) + 1(1) \Rightarrow 4 + 5 + 1 \Rightarrow 10$ (inverse exists)

$$\text{adj}A = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}^t = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj}A) = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

P.T.O.

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$A^t X = B$$

$$\Rightarrow (A^t)^{-1} A^t X = (A^t)^{-1} B$$

$$\Rightarrow IX = (A^t)^{-1} B$$

$$\Rightarrow X = (A^{-1})^t B \qquad \because (A^t)^{-1} = (A^{-1})^t$$

$$X = (A^{-1})^t B = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}^t \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \cancel{\frac{1}{10}} \begin{bmatrix} \cancel{16+0+4} \\ \cancel{-20+0+10} \\ \cancel{4+0+6} \end{bmatrix} = \cancel{\frac{1}{10}} \begin{bmatrix} \cancel{20} \\ \\ \\ \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 16+0+2 \\ 8+0-4 \\ 8+0+6 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 18 \\ 4 \\ 14 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9/5 \\ 2/5 \\ 7/5 \end{bmatrix}$$

$x = \frac{9}{5}$, $y = \frac{2}{5}$, $z = \frac{7}{5}$ Ans

EXTRA QUESTION

Q $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$

Solve
$$x - y + z = 4$$
$$x - 2y - 2z = 9$$
$$2x + y + 3z = 1$$

Sol. $\begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = CA$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$AX = B$$

$$\Rightarrow A^{-1}AX = A^{-1}B$$

$$\Rightarrow IX = A^{-1}B$$

$$\Rightarrow X = A^{-1}B$$

$$\cancel{\Rightarrow X = \frac{1}{8}\begin{bmatrix} -4 \\ -7 \\ -5 \end{bmatrix}}$$

$$CA = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\Rightarrow CA = 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \left(\frac{1}{8}C\right)A = I$$

$$\Rightarrow A^{-1} = \frac{1}{8}C \qquad \because AB = I = BA \Leftrightarrow A^{-1} = B$$

$$X = A^{-1}B = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$x = 3,\ y = -2,\ z = -1$ Ans

29|4|23

★ If $|A| = 0$

(ii) $(adj\,A)\,B \neq 0 \Rightarrow$ No solution

(iii) $(adj\,A)\,B = 0 \Rightarrow$ Infinite solutions

★ EXTRA QUESTIONS

Q1.) Solve

$x + y + z = 6$

$x + 2y + 3z = 14$

$x + 4y + 7z = 30$

Sol.
$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}}$$

$$adj\,A = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 6 & -3 \\ 1 & -2 & 1 \end{bmatrix}^{t} = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 6 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$|A| = 1(2) - 1(4) + 1(2)$

$\Rightarrow |A| = 2 - 4 + 2$

$\Rightarrow |A| = 0$

$$(adj\,A)B = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 6 & -2 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix} = \begin{bmatrix} 12 - 42 + 30 \\ -24 + 84 - 60 \\ 12 - 24 + 30 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ ~~Infinite solutions~~

Put $z = k$ in first and second equation,

$x + y = 6 - k$

$x + 2y = 14 - 3k$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6-k \\ 14-3k \end{bmatrix}$$

$$A \quad X = B$$

$\Rightarrow (A^{-1}A)X = A^{-1}B$

$\Rightarrow IX = A^{-1}B$

$\Rightarrow X = A^{-1}B$

$|A| \Rightarrow 2-1 \Rightarrow 1 \neq 0$

$$\text{adj } A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 6-k \\ 14-3k \end{bmatrix} = \begin{bmatrix} 12-2k-14+3k \\ -6+k+14-3k \end{bmatrix} = \begin{bmatrix} k-2 \\ 8-2k \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k-2 \\ 8-2k \end{bmatrix}$$

$x = k-2$

$y = 8-2k$

$z = k$

Putting these values in third equation,

$(k-2) + 4(8-2k) + 7k = 30$

$\Rightarrow k - 2 + 32 - 8k + 7k = 30$

$\Rightarrow \cancel{30} - \cancel{8k} + \cancel{8k} = \cancel{30}$ Equation satisfied

$\Rightarrow 0$ $\quad \therefore$ Infinite solutions

Q2.) Solve $3x - y - 2z = 2$, $2y - z = -1$, $3x - 5y = 3$.

Sol.
$$\begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$A \quad X \quad B$$

$A_{11} = -5 \quad A_{12} = -3 \quad A_{13} = -6$

$A_{21} = 10 \quad A_{22} = 6 \quad A_{23} = 12$

$A_{31} = 5 \quad A_{32} = 3 \quad A_{33} = 6$

$$adj\,A = \begin{bmatrix} -5 & -3 & -6 \\ 10 & 6 & 12 \\ 5 & 3 & 6 \end{bmatrix}^t = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}$$

$$|A| \Rightarrow (-5)(3) + (-3)(-1) + (-6)(-2) \Rightarrow -15 + 3 + 12 \Rightarrow 0$$

$$(adj\,A)B \Rightarrow \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} -10-10+15 \\ -6-6+9 \\ -12-12+18 \end{bmatrix} \Rightarrow \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix}$$

$(adj\,A)B \neq 0$

$\therefore$ No solution

Q3.) The sum of three numbers is 2. If twice the second number is added to the sum of first and third, the sum is 1. By adding second and third to 5 times the first number, we get 6. Find the numbers using matrix method.

Sol. Let the three numbers be x, y and z

A.T.Q.

$$x+y+z=2$$
$$x+z+2y=1$$
$$y+z+5x=6$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 5 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$$

$$AX = B$$
$$\Rightarrow (A^{-1}A)X = A^{-1}B$$
$$\Rightarrow IX = A^{-1}B$$
$$\Rightarrow X = A^{-1}B$$

$A_{11} = 1$ $\quad A_{12} = 4$ $\quad A_{13} = -9$

$A_{21} = 0$ $\quad A_{22} = -4$ $\quad A_{23} = 4$

$A_{31} = -1$ $\quad A_{32} = 0$ $\quad A_{33} = 1$

$|A| \Rightarrow 1(1) + 4(1) + (-9)(1) \Rightarrow 1+4-9 \Rightarrow -4$ (inverse exists)

$$\text{adj } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & -9 \\ 0 & -4 & 4 \\ -1 & 0 & 1 \end{bmatrix}^t = \begin{bmatrix} 1 & 0 & -1 \\ 4 & -4 & 0 \\ -9 & 4 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{-4}\begin{bmatrix} 1 & 0 & -1 \\ 4 & -4 & 0 \\ -9 & 4 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 \\ -4 & 4 & 0 \\ 9 & -4 & -1 \end{bmatrix}$$

$$X = A^{-1}B$$

$$X = \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 \\ -4 & 4 & 0 \\ 9 & -4 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2+0+6 \\ -8+4+0 \\ 18-4-6 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ -4 \\ 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$x = 1, \; y = -1, \; z = 2$ Ans

30/4/23

☆ MISCELLANEOUS EXERCISES ON CHAPTER 4

Q1) Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ.

Sol. $A_{11} = -x^2 - 1$

$A_{12} = -(\cancel{-\sin} - x\sin\theta - \cos\theta) = x\sin\theta + \cos\theta$

$A_{13} = x\cos\theta - \sin\theta$

$|A| = x(-x^2-1) + \sin\theta(x\sin\theta + \cos\theta) + \cos\theta(x\cos\theta - \sin\theta)$

$\Rightarrow |A| = -x^3 - x + x\sin^2\theta + \cancel{\sin\theta\cos\theta} + x\cos^2\theta - \cancel{\sin\theta\cos\theta}$

$\Rightarrow |A| = -x^3 - x + x(\sin^2\theta + \cos^2\theta)$

$\Rightarrow |A| = -x^3 \cancel{-x} \cancel{+x}$

$\Rightarrow |A| = -x^3$

$|A|$ is independent of θ

Hence, Proved

Q2) Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Sol. $A_{11} = \cos\alpha\cos\beta$

$A_{12} = \sin\beta\cos\alpha$

$A_{13} = -\sin\alpha\sin^2\beta - \sin\alpha\cos^2\beta = -\sin\alpha(\sin^2\beta + \cos^2\beta) = -\sin\alpha$

$|A| = \cos\alpha\cos\beta(\cos\alpha\cos\beta) + \cos\alpha\sin\beta(\sin\beta\cos\alpha) + (-\sin\alpha)(-\sin\alpha)$

$\Rightarrow |A| = \cos^2\alpha\cos^2\beta + \sin^2\beta\cos^2\alpha + \sin^2\alpha$

$\Rightarrow |A| = \cos^2\alpha(\cos^2\beta + \sin^2\beta) + \sin^2\alpha$

$\Rightarrow |A| = \cos^2\alpha + \sin^2\alpha$

$\Rightarrow |A| = 1$ Ans

Q3.) If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$

Sol. $B_{11} = 3$ $\quad B_{12} = 1$ $\quad B_{13} = 2$

$B_{21} = 2$ $\quad B_{22} = 1$ $\quad B_{23} = 2$

$B_{31} = 6$ $\quad B_{32} = 2$ $\quad B_{33} = 5$

$|B| \Rightarrow 3 + 2 - 4 = 1$ (inverse exists)

$$\text{adj} B = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}^t = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \text{adj} B = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

$$\Rightarrow (AB)^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\Rightarrow (AB)^{-1} = \begin{bmatrix} 9 - 30 + 30 & -3 + 12 - 12 & 3 - 10 + 12 \\ 3 - 15 + 10 & -1 + 6 - 4 & 1 - 5 + 4 \\ 6 - 30 + 25 & -2 + \overset{12}{\cancel{30}} - 10 & 2 - \overset{10}{\cancel{25}} + 10 \end{bmatrix}$$

$$\Rightarrow (AB)^{-1} = \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \text{ Ans}$$

Q4.) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$. Verify that

i) $[\text{adj } A]^{-1} = \text{adj}(A^{-1})$

ii) $(A^{-1})^{-1} = A$

Sol. $A_{11} = 14$ $\quad A_{12} = -9$ $\quad A_{13} = -1$

$A_{21} = -9$ $\quad A_{22} = 4$ $\quad A_{23} = 1$

$A_{31} = -1$ $\quad A_{32} = 1$ $\quad A_{33} = -1$

$|A| = 14 - 18 - 1 \Rightarrow -5$ (inverse exists)

$$\text{adj } A = \begin{bmatrix} 14 & -9 & -1 \\ -9 & 4 & 1 \\ -1 & 1 & -1 \end{bmatrix}^t = \begin{bmatrix} 14 & -9 & -1 \\ -9 & 4 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{-5}\begin{bmatrix} 14 & -9 & -1 \\ -9 & 4 & 1 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -14/5 & 9/5 & 1/5 \\ 9/5 & -4/5 & -1/5 \\ 1/5 & -1/5 & 1/5 \end{bmatrix}$$

(i) L.H.S.

$(\text{adj } A)_{11} = -5$ $\quad (\text{adj } A)_{12} = -10$ $\quad (\text{adj } A)_{13} = -5$

$(\text{adj } A)_{21} = -10$ $\quad (\text{adj } A)_{22} = -15$ $\quad (\text{adj } A)_{23} = -5$

$(\text{adj } A)_{31} = -5$ $\quad (\text{adj } A)_{32} = -5$ $\quad (\text{adj } A)_{33} = -25$

$|\text{adj } A| \Rightarrow 14(-5) + (-9)(-10) + (-1)(-5) \Rightarrow -70 + 90 + 5 \Rightarrow 25$ (inverse exists)

$$\text{adj}(\text{adj } A) = \begin{bmatrix} -5 & -10 & -5 \\ -10 & -15 & -5 \\ -5 & -5 & -25 \end{bmatrix}^t = \begin{bmatrix} -5 & -10 & -5 \\ -10 & -15 & -5 \\ -5 & -5 & -25 \end{bmatrix}$$

$$[\text{adj } A]^{-1} = \frac{1}{|\text{adj } A|}[\text{adj}(\text{adj } A)] = \frac{1}{25}\begin{bmatrix} -5 & -10 & -5 \\ -10 & -15 & -5 \\ -5 & -5 & -25 \end{bmatrix} = \begin{bmatrix} -1/5 & -2/5 & -1/5 \\ -2/5 & -3/5 & -1/5 \\ -1/5 & -1/5 & -1 \end{bmatrix}$$

R.H.S

$(A^{-1})_{11} = -1/5$ $\quad (A^{-1})_{12} = -2/5$ $\quad (A^{-1})_{13} = -1/5$

$(A^{-1})_{21} = -2/5$ $\quad (A^{-1})_{22} = -3/5$ $\quad (A^{-1})_{23} = -1/5$

$(A^{-1})_{31} = -1/5$ $\quad (A^{-1})_{32} = -1/5$ $\quad (A^{-1})_{33} = -1$

$$|A^{-1}| \Rightarrow \frac{14}{25} - \frac{18}{25} - \frac{1}{25} \Rightarrow \frac{-5}{25} \Rightarrow \frac{-1}{5}$$

$$\text{adj}(A^{-1}) = \begin{bmatrix} -1/5 & -2/5 & -1/5 \\ -2/5 & -3/5 & -1/5 \\ -1/5 & -1/5 & -1 \end{bmatrix}^t = \begin{bmatrix} -1/5 & -2/5 & -1/5 \\ -2/5 & -3/5 & -1/5 \\ -1/5 & -1/5 & -1 \end{bmatrix}$$

L.H.S. = R.H.S.

Hence, verified

(ii) $$(A^{-1})^{-1} = \frac{1}{|A^{-1}|}\left[\text{adj}(A^{-1})\right] = \frac{1}{-1/5}\begin{bmatrix} -1/5 & -2/5 & -1/5 \\ -2/5 & -3/5 & -1/5 \\ -1/5 & -1/5 & -1 \end{bmatrix}$$

$$\Rightarrow (A^{-1})^{-1} = -5\begin{bmatrix} -1/5 & -2/5 & -1/5 \\ -2/5 & -3/5 & -1/5 \\ -1/5 & -1/5 & -1 \end{bmatrix}$$

$$\Rightarrow (A^{-1})^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = A$$

Hence, verified-

Q5.) Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

Sol. $A_{11} = y(x+y) - x^2$

$A_{12} = x(x+y) - y^2$

$A_{13} = xy - (x+y)^2$

$$|A| = xy(x+y) - x^3 + xy(x+y) - y^3 + xy(x+y) - (x+y)^3$$

$$\Rightarrow |A| = \cancel{x^2y} + \cancel{xy^2} - x^3 + \cancel{x^2y} + \cancel{xy^2} - y^3 + \cancel{x^2y} + \cancel{xy^2} - x^3 - y^3 - \cancel{3x^2y} - \cancel{3xy^2}$$

$$\Rightarrow |A| = -2x^3 - 2y^2$$

$$\Rightarrow |A| = -2(x^3 + y^3) \quad \underline{\text{Ans}}$$

Q6) Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Sol. $A_{11} \Rightarrow (x+y)^2 - xy$

$A_{12} \Rightarrow -[(x+y)-y] \Rightarrow y-(x+y) \Rightarrow \cancel{y}-x-\cancel{y} \Rightarrow -x$

$A_{13} \Rightarrow x-\cancel{x}-y \Rightarrow -y$

$|A| = (x+y)^2 - xy - x^2 - y^2$

$\Rightarrow |A| = \cancel{x^2}+\cancel{y^2}+2xy - xy - \cancel{x^2} - \cancel{y^2}$

$\Rightarrow |A| = xy$ Ans.

Q7) Solve the system of equations

$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$

$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$

$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$

Sol. $\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix} \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

$AX = B$

$\Rightarrow (A^{-1}A)\cdot X = A^{-1}B$

$\Rightarrow IX = A^{-1}B$

$\Rightarrow X = A^{-1}B$

$A_{11} = 75$ $\quad A_{12} = 110$ $\quad A_{13} = 72$

$A_{21} = 150$ $\quad A_{22} = \cancel{20}-100$ $\quad A_{23} = 0$

$A_{31} = +75$ $\quad A_{32} = 30$ $\quad A_{33} = -24$

$|A| \Rightarrow 2(75) + 3(110) + 10(72) \Rightarrow 150 + 330 + 720 \Rightarrow 1200$ (inverse exists)

$$\text{adj } A = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}^t = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/5 \end{bmatrix}$$

$$\left.\begin{aligned} \frac{1}{x} &= \frac{1}{2} \Rightarrow x = 2 \\ \frac{1}{y} &= \frac{1}{3} \Rightarrow y = 3 \\ \frac{1}{z} &= \frac{1}{5} \Rightarrow z = 5 \end{aligned}\right\} \text{Ans}$$

- Choose the correct answer in Exercises 8 and 9.

Q8.) If x, y, z are non-zero real numbers, then the inverse of matrix $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is

(A) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ (B) $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

(C) $\frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ (D) $\frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Sol $A_{11} = yz$ $\quad A_{12} = 0$ $\quad A_{13} = 0$

$A_{21} = 0$ $\quad A_{22} = xz$ $\quad A_{23} = 0$

$A_{31} = 0$ $\quad A_{32} = 0$ $\quad A_{33} = xy$

$|A| \Rightarrow xyz$ (inverse exists)

$$\text{adj } A = \cancel{\frac{1}{|A|}} \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}^t = \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{xyz} \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix} = \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

$$\therefore (A) \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$ Ans

---

Q9.) Let $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \le \theta \le 2\pi$. Then

(A) Det $(A) = 0$ $\qquad$ (B) Det $(A) \in (2, \infty)$

(C) Det $(A) \in (2, 4)$ $\qquad$ (D) Det $(A) \in [2, 4]$

Sol. $A_{11} = \sin^2\theta + 1$

$A_{12} = \cancel{2\sin\theta}\ 0$

$A_{13} = \sin^2\theta + 1$

$|A| = 2 + 2\sin^2\theta$

$0 \le \theta \le 2\pi$

$\Rightarrow -1 \le \sin\theta \le 1$

$\Rightarrow 0 \le \sin^2\theta \le 1$ [$\because \sin^2\theta$ can never be negative]

$\Rightarrow 2 \times 0 \le 2 \times \sin^2\theta \le 2 \times 1$

$\Rightarrow 0 \le 2\sin^2\theta \le 2$

$\Rightarrow 0+2 \le 2\sin^2\theta + 2 \le 2+2$

$\Rightarrow 2 \le 2\sin^2\theta + 2 \le 4$

$\Rightarrow 2 \le |A| \le 4$

$\therefore$ (D) $\det(A) \in [2, 4]$ Ans